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A NOTE ON MAGIC GRAPHS

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A note on magic graphs

In [1] Stewart defined for a finite graph G not containing isolated vertices the spaces S(G) and Z(G); S(G) is the space of all real-valued functions f defined on the set of edges E(G) of G with the property that $\sum\{f(e) \mid e \text{ is incident with } v\} =: \sigma_v(f)$ is independent of the vertex v in G, and Z(G) is the subspace of S(G) consisting of the functions f with $\sigma(f) = \sigma_v(f) = 0$.
He proved that if G is connected then

(1) $E - n + 1 \le \dim S(G) \le E - n + 2$

(2) $\dim Z(G) \le \dim S(G) \le 1 + \dim Z(G)$

where E is the number of edges and n the number of vertices of G, but he was unable to determine the exact values of dim S(G) and dim Z(G).
In this note I shall prove:

Theorem 1. If G is connected then $\dim S(G) = E - n + 2$ iff the vertices of G can be coloured with blue and red in such a way that no two vertices of the same colour are adjacent and, moreover, the number of blue vertices equals the number of red vertices.
In other words: $\dim S(G) = E - n + 2$ if G is bipartite in two sets of equal cardinality
and $\dim S(G) = E - n + 1$ otherwise.

Theorem 2. If G is connected then
$\dim Z(G) = \dim S(G) - 1 = E - n$ if G contains a circuit of odd length
and $\dim Z(G) = E - n + 1$ otherwise.

Corollary. Call a graph G semimagic if $\dim S(G) > \dim Z(G)$; then we have: $K_{n,m}$ is semimagic iff $n = m$; K_n is semimagic for all $n \ge 2$.

Theorem 3. Let G consist of the components G_i $(1 \le i \le \tau(G))$, where $\tau(G)$ is the number of components of G.
Then $\dim Z(G) = \sum \dim Z(G_i)$, and if $\delta_i = \dim S(G_i) - \dim Z(G_i)$ then $\dim S(G) = \sum \dim Z(G_i) + \Pi\delta_i = \sum \dim S(G_i) - \sum\delta_i + \Pi\delta_i$.
In particular if for all i $S(G_i) \neq Z(G_i)$ then $\dim S(G) = \sum \dim S(G_i) - \tau(G) + 1$ and if $S(G_i) = Z(G_i)$ for

$\tau'(G) > 0$ components of G then
$\dim S(G) = \sum \dim S(G_i) - \tau(G) + \tau'(G)$.

(This is obvious, but Stewart gives the incorrect result
$\dim S(G) = 1 - \tau(G) + \sum \dim S(G_i)$ if for all i $\dim S(G_i) > 0$.
A counterexample to this is the graph

).

Proof.

Let G be connected, $\dim S(G) = E - n + \delta$, $\dim Z(G) = E - n + \zeta$ and let $f \in S(G)$.
The proof is with induction on E, the number of edges of G.
We distinguish several cases:

(A) G contains a circuit of even length $C = (v_0, v_1, \ldots, v_{2k-1})$.
Let G' be the graph obtained from G by deleting the edge $v_0 v_{2k-1}$.
G' is connected, and G' is bipartite in equal parts iff G is and G' contains an odd circuit iff G does.
Define f' by $f'(v_i v_{i+1}) = f(v_i v_{i+1}) + (-1)^i f(v_0 v_{2k-1})$ $(i=0,\ldots,2k-2)$

and $f'(e) = f(e)$ if $e \notin C$.

Then $f' \in S(G')$ and $\sigma(f') = \sigma(f)$.
Conversely if f' on G' is given, then f on G can be constructed by

$$\begin{cases} f(v_0 v_{2k-1}) = x \\ f(v_i v_{i+1}) = f'(v_i v_{i+1}) - (-1)^i x \\ f(e) = f'(e) & \text{if } e \notin C. \end{cases}$$

Since x is arbitrary this proves $\dim S(G') = \dim S(G) - 1$ and $\dim Z(G') = \dim Z(G) - 1$. Since $n' = n$ and $E' = E - 1$ it follows that $\delta' = \delta$ and $\zeta' = \zeta$, so the theorems are valid for G if and only if they are valid for G'.

(B) G does not contain a circuit of even length, but contains two circuits of odd length:

$$C_1 = (v_0, v_1, \ldots, v_{2k}) \quad \text{and} \quad C_2 = (w_0, w_1, \ldots, w_{2l}).$$

Since G does not contain a circuit of even length, C_1 and C_2 do not have common edges.

Since G is connected, C_1 and C_2 are connected by a way $W = (u_0, u_1, \ldots, u_s)$ where

$$W \cap C_1 = \{u_0\} \quad \text{and} \quad W \cap C_2 = \{u_s\} \quad \text{and possibly } s = 0.$$

Let $u_0 = v_0$, $u_s = w_0$, $w_{2l+1} := w_0$.

Let G' be the graph obtained from G by deleting the edge $v_0 v_{2k}$.

As above an f' can be defined by

$$\begin{cases} f'(e) = f(e) \quad \text{for } e \notin C_1 \cup C_2 \cup W \\ f'(v_i v_{i+1}) = f(v_i v_{i+1}) - (-1)^i \, f(v_0 v_{2k}) \\ f'(u_j u_{j+1}) = f(u_j u_{j+1}) + 2(-1)^j \, f(v_0 v_{2k}) \\ f'(w_j w_{j+1}) = f(w_j w_{j+1}) + (-1)^j \, (-1)^s \, f(v_0 v_{2k}) \end{cases}$$

and again it follows that $\delta' = \delta$ and $\zeta' = \zeta$.

(C) G contains one circuit of odd length: $C = (v_0, v_1, \ldots, v_{2k})$. Define $v_{2k+i} := v_{i-1}$. Here $E = n$, so we have to prove $\delta = 1$ and $\zeta = 0$.

Fix a $\sigma \in \mathbb{R}$; then an $f \in S(G)$ with $\sigma(f) = \sigma$ can be defined in one and only one way:

Each v_i is the root of a (possibly empty) tree on which f is completely determined.

To satisfy the conditions $\sigma_{v_i}(f) = \sigma$ we get 2k+1 equations

$f(v_{i-1} v_i) + f(v_i v_{i+1}) = a_i$ $(1 \le i \le 2k+1)$ with the unique solution

$$f(v_{i-1} v_i) = \frac{1}{2} a_{i-1} + \frac{1}{2} \sum_{j=0}^{2k-1} (-1)^j \, a_{i+j} \quad (1 \le i \le 2k+1) \text{ (where } a_{i+2k+1} = a_i).$$

This proves both theorems for graphs which contain a circuit of odd length.

(D) G contains no circuit, i.e. is a tree.

Fix a root v_0 of G and a $\sigma \in R$, then there is a unique f such that $\sigma_v(f) = \sigma$ for $v \neq v_0$.

Now if $\sigma_{v_0}(f) = \sigma$ then $\dim S(G) = 1$ else $\dim S(G) = 0$, and in either case $\dim Z(G) = 0$.

Since $E = n-1$ and $\dim Z(G) = 0$, we have $\zeta = 1$, which proves theorem 2.

G is connected and does not contain a circuit of odd length, hence G is bipartite in a unique way: $G = G_1 \cup G_2$. Now if $\dim S(G) = 1$ and $\sigma(f) \neq 0$ then $|G_1| = |G_2|$ since

$\sigma \cdot |G_1| = \sum_e f(e) = \sigma \cdot |G_2|$.

Conversely, if $|G_1| = |G_2|$ and $\sigma_v(f) = \sigma$ for $v \neq v_0$ then

$\sigma_{v_0}(f) = \sum_e f(e) - \sigma \cdot (|G_1| - 1) = \sigma$.

Therefore if $|G_1| = |G_2|$ then $\delta = 2$ else $\delta = 1$. This proves everything.

Reference

1. B.M. Stewart, Magic graphs, Can. J. Math., 18 (1966), 1031-1059.